J. Fluid Mech. (1971), *vol.* 47, *part* 4, *pp.* 689–711
Printed in Great Britain

The spin down of rotating stratified fluids

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(Received 1 July 1970)

The transient process by which an incompressible dissipative rotating stratified fluid adjusts to a small change in the rotation rate of its container is examined theoretically. The aim is to clarify the effects of the imposed density stratification and of the boundary condition specified for the density perturbation on the behaviour of the fluid, particularly during the time span when the adjustment is performed in a homogeneous fluid. For a weakly stratified fluid in a cylinder, it is shown how these two factors govern the nature and intensity of boundary layers on the vertical wall which close the secondary meridional circulation generated by Ekman layers along the horizontal boundaries. For a more strongly stratified fluid, the usefulness and importance of potential vorticity conservation in determining the quasi-steady motion is verified, and a calculation for a spherical container demonstrates some new features that arise only when the container boundaries are not normal or parallel to the rotation axis. It is shown that experimental results of Holton (1965) are in less good agreement with predictions of the linear theory than had been previously indicated.

1. Introduction

If a container rotating with uniform angular velocity about a fixed axis and filled with an incompressible dissipative fluid has its rotation rate increased impulsively, the transient process by which the fluid adjusts to the new angular velocity of the container is known as 'spin up'. For a slightly viscous fluid of uniform density, Greenspan & Howard (1963) demonstrated that the adjustment is controlled primarily by Ekman boundary layers on the container surface. By establishing a secondary circulation throughout the fluid which transports angular momentum, the layers restore the dominant interior motion to rigid rotation much more rapidly than the time that would be required by viscous diffusion acting alone. A very interesting variant of this mechanism was described by Bretherton & Spiegel (1968) in their model for an astrophysical application of spin up (a thorough extension of their ideas was presented by Kroll & Veronis (1970)).

The purpose of this paper is to elucidate the dynamics of spin up in a rotating density-stratified fluid when the relative change in rotation rate is infinitesimally small. In particular we shall strive to understand the mechanisms by which the perturbation density boundary condition and the mean density stratification influence motions in and away from the boundary layers. These questions have been considered previously in a sequence of papers commencing with one by

Holton (1965), but it is fair to say that at least two aspects of the fluid dynamics could be clarified by some additional description. Because the motions in the spin up of a homogeneous fluid are so different from those of a stratified one, it appeared worthwhile to investigate in detail the connexion between these cases by studying the spin up problem as a function of the ratio between the inverse rotation rate and the characteristic time scale of the mean stratification. By this means the transition from dynamics dominated by rotation to those characteristic of the interaction of rotation and stratification can be observed. Such a procedure was adopted by Barcilon & Pedlosky (1967*b*) for certain steady motions in rotating stratified fluids and was implied in some brief remarks by Walin (1969) for spin up. We use a cylindrical container and concentrate on the region near the vertical boundary wall where the effects of the transition are most substantial, i.e. where the influence of the stratification on the largest motions in the fluid first occurs as the stratification is increased from zero. Although solutions have been obtained for all values of the stratification parameter within the transition régime, they are presented here only for a specific range wherein it is readily apparent how the boundary condition on the density perturbation influences the motions. The second feature of spin-up dynamics concerns the central role of potential vorticity conservation when the time scales of the rotation and the stratification are comparable. We show that the simplifications arising from fully exploiting this property are very helpful in understanding the flow in a cylinder and are virtually essential for other container geometries. The only non-cylindrical container we consider is spherical, but it is clear that the boundary-layer coupling which arises there is common to all situations in which the normal to the container boundary is neither parallel nor perpendicular to the rotation axis (and the direction of gravity).

All effects resulting from the finite amplitude of the perturbation in rotation rate are ignored, and in particular we choose to consider the then equivalent problem of the 'spin down' of fluid initially rotating faster than its container. We also neglect the inevitable distortion of the mean stratification by the rotation, thereby assuming that even for a density diffusive fluid all motions relative to the basic rotation are strictly due to the perturbation. Although this model could conceivably account for the results of actual spin-up observations, the purpose of this paper is primarily to clarify the properties of the model rather than to decide conclusively whether its equations are adequate to predict the laboratory behaviour of rotating stratified fluids. Evidence for the realizeability of the linear results has been both ambiguous (Holton (1965) where experimental conditions only marginally fulfilled the requirements for the validity of the model) and downright discouraging (Greenspan 1968, chapter 6). Our only contribution to this question is a brief comparison between the currently published experimental data and conclusions of two versions of the linear theory.†

The initial conditions of the spin-down problem represent a member of the class of infinitesimal amplitude flows which in a perfectly non-dissipative rotating stratified fluid are steady in time. The methods used in this paper are in

† Careful experiments by Buzyna & Veronis (to be published in *J. Fluid Mech.*) indicate that the linear description can be appropriate in laboratory conditions.

fact suitable for studying the time variation produced by dissipation of any such flow in a container of any shape. We confine ourselves here to spin down in familiar geometries because of the great analytical simplifications, the property that an initial change in rotation rate is also a member of the corresponding class of flows in a homogeneous fluid, and the considerable previous interest in this problem.

2. Formulation

We wish to consider motions of a viscous, heat-conducting fluid rotating with uniform angular velocity Ω and stratified under a gravitational force g anti-parallel to the vertical rotation vector. The motions are assumed to satisfy the Boussinesq approximation to the hydrodynamic equations and to deviate very slightly from a time-independent state of rigid rotation and hydrostatic equilibrium. Now such a basic state is forbidden in a thermally diffusive medium, but it is permissible if we assume that the centrifugal curvature of the mean iso-density lines is negligible. This condition, which may be stated as $\Omega^2 H g^{-1} = F_R \ll 1$, where H is a representative length scale of the container, is consistent with a constant vertical gradient $\Delta\rho$ of mean density. The perturbation motion is generated by impulsively changing the angular velocity of the container, which for the most part is assumed cylindrical to take advantage both of previous interest in this geometry and of obvious simplifications.

If we introduce scales H, Ω^{-1}, and $\epsilon\Omega^{-1}H$ for distance $\mathbf{r}$, time t, and velocity $\mathbf{u}$ of the perturbation motion, we may easily obtain corresponding scales for the density ρ, temperature T, and pressure p in a fluid of average density ρ_0. The dimensionless equations of motion under the foregoing approximations and with respect to a rotating co-ordinate system have been described in detail by Greenspan (1968). Neglecting non-linear terms by the additional requirement $\epsilon \ll 1$, we write those equations in the following way:

$$\partial\mathbf{u}/\partial t + 2\hat{\mathbf{k}} \times \mathbf{u} + \nabla p - T\hat{\mathbf{k}} = E\nabla^2\mathbf{u}, \tag{2.1}$$

$$\partial T/\partial t + 4S\hat{\mathbf{k}}.\mathbf{u} = \sigma^{-1}E\nabla^2 T, \tag{2.2}$$

$$\nabla.\mathbf{u} = 0, \tag{2.3}$$

in which ρ has been eliminated by the equation of state $\rho = -T$; $\hat{\mathbf{k}}$ is a unit vertical vector; $E = \nu\Omega^{-1}H^2$ is the Ekman number for kinematic viscosity ν; σ is the Prandtl number; and $S = (\Delta\rho)\rho_0^{-1}g(2\Omega)^{-2}H^{-1}$ is the square of the ratio of the Brunt–Väisälä frequency to twice the rotation rate. The parameter E is assumed very small compared to one, σ is at least $O(1)$ with respect to E, and S is no larger than $O(1)$.

If we refer the perturbations to a co-ordinate system rotating with the slightly altered angular velocity and employ cylindrical co-ordinates (r, ϕ, z) with corresponding velocity components (u, v, w), we can impose the initial conditions

$$\mathbf{u}(t=0) = r\hat{\boldsymbol{\phi}}, \quad T(t=0) = 0, \tag{2.4}$$

and either set of boundary conditions

$$\mathbf{u} = T = 0 \quad \text{on} \quad \Sigma, \tag{2.5}$$

$$\mathbf{u} = \hat{\mathbf{n}}.\nabla T = 0 \quad \text{on} \quad \Sigma, \tag{2.6}$$

where Σ is the surface of the circular cylinder

$$\{z = \tfrac{1}{2} \pm \tfrac{1}{2},\ 0 \leqslant r \leqslant A;\ 0 \leqslant z \leqslant 1,\ r = A\}$$

and $\hat{\mathbf{n}}$ is the exterior unit normal vector. We denote by problem I or II the system, (2.1)–(2.4) subject to (2.5) or (2.6), corresponding to zero perturbation temperature or heat flux on the rigid walls. For the linearized problem we may assume the motion remains independent of the azimuthal angle ϕ and thereby introduce a stream function ψ so that

$$u = \partial\psi/\partial z, \quad w = -r^{-1}\,\partial(r\psi)/\partial r.$$

Certain fundamental properties of the fluid motion following the initial disturbance greatly simplify the determination of the decay process. In particular, if the fluid were strictly non-dissipative, the motion represented by (2.4) would be steady for later times because it is an allowed *geostrophic* flow, satisfying time-independent versions of (2.1)–(2.3) with zero Ekman number and the appropriate inviscid boundary condition. The motion thus owes its time variation entirely to the presence of the very small but non-negligible dissipation, and although there is a fairly extensive collection of initial disturbances which do possess this property (Howard & Siegmann 1969), it is easy to imagine simple initial perturbations which do not. Greenspan (1968, e.g. p. 56) has indicated the rather severe analytical difficulties which arise in the solutions of such problems for a homogeneous fluid ($S = 0$).

Another important feature is evident if we consider the evolution of (2.4) on dimensionless time scales much shorter than the dimensionless time scale E^{-1} characterizing diffusion over a distance the size of the container. As might be anticipated, on sufficiently short time scales dissipation will induce only asymptotically small corrections to the flow except in certain asymptotically thin regions adjacent to the boundaries. This assumption underlies the analysis here and all previous studies of spin down. At any time we refer to that part of the fluid away from these boundary regions as the interior region. In order to determine the most interesting aspects of the decay process, it is germane to ascertain the earliest time scale τ shorter than E^{-1} on which the interior motion is significantly altered (if one exists!), the structures of the boundary regions and the motions therein on this time scale, and the physical process by which the interior flow is changed.

In a homogeneous fluid, Greenspan & Howard (1963) demonstrated that Ekman boundary layers, which form with thickness $O(E^{\frac{1}{2}})$ on the horizontal surfaces in the dimensionless time $t = O(1)$, dissipate the imposed vorticity in the interior during the time when $\tau = E^{\frac{1}{2}}t = O(1)$. The Ekman layers relax the mean vortex lines by expelling an $O(E^{\frac{1}{2}})$ flux, so that their effect on the interior velocity fields during the time scale $\tau = O(1)$ may be expressed by the compatibility condition

$$w = \pm\tfrac{1}{2}E^{\frac{1}{2}}\hat{\mathbf{k}}.\nabla\times\mathbf{u} \quad \text{on} \quad z = \tfrac{1}{2}\pm\tfrac{1}{2}. \tag{2.7}$$

Using this relation an adequate picture of spin down may be obtained by just determining the τ-scale evolution of the significant motions in the interior and in the other boundary regions which, unlike the Ekman layers, do not have controlling influences on the interior flow. This procedure essentially ignores the detailed development of the small motions in the interior and the substantial motions in the boundary regions on time scales shorter than τ. It is important to realize that the description of spin down on the dynamically interesting time scale is not impaired by this simplification, and all the motions on the intervening time scales could be included but with a great loss of clarity (Greenspan 1968, p. 41). It is also true, however, that certain developments in the boundary regions on time scales shorter than τ possess somewhat unexpected features (Barcilon 1968).

Now the structure of the Ekman layer depends on a balance of Coriolis and viscous forces acting on the horizontal velocity components and is essentially unchanged in an inhomogeneous fluid. Thus except for fluids with extremely rapid thermal diffusion or with exceedingly strong stratification which are not considered here, the Ekman layers can form and we are free to apply (2.7) in a stratified fluid also. Indeed, Ekman layers *must* form in order to satisfy the no slip boundary condition on the azimuthal velocity, because on short time scales any height variation of the main interior velocity component is forbidden and the other allowable boundary layer on the horizontal walls (a diffusion layer with thickness $O(t^{\frac{1}{2}}E^{\frac{1}{2}})$) is only capable of satisfying a temperature boundary condition. The presence of Ekman layers therefore demands that the initial interior vorticity is in some way modified when $\tau = E^{\frac{1}{2}}t = O(1)$, because from (2.7), (2.3) and (2.1) with $\partial/\partial t = O(E^{\frac{1}{2}})$, it is clear that this is the *shortest* time on which the interior can support both the $O(E^{\frac{1}{2}})$ flux injected into it by the layers and a time variation in the $O(1)$ velocity.

In analogy with the case of a homogeneous fluid, we therefore plan to focus attention on developments during the τ time scale. We reiterate that the primacy of this time scale would be displaced if the container were mechanically stress free and thus would suppress strong Ekman layer formation, or if the thermal properties of the fluid were to violate the conditions $\sigma \geqslant O(1)$, $S \leqslant O(1)$ and would thereby introduce a significantly shorter time scale. In addition, we emphasize that the successful study of the spin-down process on the τ time scale by neglecting details of the motions on earlier time scales relies on the fundamental assumptions of a model with negligibly small parameters ϵ and F_R. Formal bounds on these parameters so that they do not upset the relevant dynamics expressed by (2.1) to (2.6) are $(\epsilon, F_R) \ll E^{\frac{1}{2}}$, and these constraints represent the most important limitations on the validity of our solution procedure. We conclude this summary of restrictions by pointing out that even using the 'linear' model with moderate stratification, considerable care may be required to demarcate the extent of the interior region on the τ time scale (Walin 1969) and to adequately describe the motions in the boundary regions.

In a stratified fluid for which $S = O(1)$, the adjustment in the interior velocity which actually occurs on the τ time scale is governed by balancing two effects of the meridional circulation induced by the Ekman layers, specifically vortex tube expansion (or decrease of relative vorticity) against departure from horizontal of the isodensity lines. This conservation mechanism is at the heart of previous spin-down analyses (Holton 1965; Pedlosky 1967; Walin 1969; Sakurai 1969*a*, *b*), and we shall discuss it further in § 4. For the present we observe that if the stratification is *weak* $S \ll O(1)$, the lowest-order dynamics of the interior region on the τ time scale are the same as in a homogeneous fluid, and we shall examine in § 3 how the motions in the other boundary regions differ. This behaviour of the time-dependent motions in a weakly stratified rotating fluid is rather striking in comparison with the results of Barcilon & Pedlosky (1967*b*) and Pedlosky (1969). They realized that for time-independent motions the lowest-order dynamics of the interior region are homogeneous only in the much more restrictive case $S \ll O(E^{\frac{1}{2}})$. Because vortex tube expansion cannot be supported by motions with very long time scales, the interior circulation required to maintain the Ekman layers will be inhibited by the stratification unless the latter constraint is satisfied.

3. A mathematical description of weakly stratified spin down

The fact that the lowest-order interior dynamics when $S \ll O(1)$ are the same as in a homogeneous fluid follows immediately after scaling (2.1) and (2.2) with $t = E^{-\frac{1}{2}}\tau$ and $\psi = E^{\frac{1}{2}}X$:

$$\left.\begin{aligned} 2V = P_r, \quad V_\tau + 2X_z = O(E^{\frac{1}{2}}), \quad P_z = T, \\ T_\tau + 4S(1/r)(Xr)_r = O(E^{\frac{1}{2}}/\sigma). \end{aligned}\right\} \qquad (3.1\,a\text{–}d)$$

Capital letters denote functions with the interior region as their domain, and subscripts represent partial derivatives. Using (2.4) and (2.7), the solution for the spin-down problem is

$$\left.\begin{aligned} V = r\,e^{-2\tau}, \quad X = r(z-\tfrac{1}{2})\,e^{-2\tau}, \\ T = O(S) = S4(z-\tfrac{1}{2})\,(1-e^{-2\tau}). \end{aligned}\right\} \qquad (3.2\,a\text{–}c)$$

The weak stratification leads to a small adjustment in the temperature field, but otherwise there is no change in the interior solution and particularly in its driving mechanism. Since the temperature boundary condition on the horizontal walls is satisfied by a diffusion layer, the remaining question is the manner in which the boundary conditions at the vertical side wall are satisfied.

In a homogeneous fluid, Greenspan & Howard (1963) showed how two Stewartson boundary layers (thicknesses $O(E^{\frac{1}{3}})$ and $O(E^{\frac{1}{4}})$) conspire together to satisfy the vanishing of the three velocity components at the side wall. Barcilon & Pedlosky (1967*b*) described the evolution of the Stewartson layers as S increases from zero and used the three resulting boundary layers which occur when $O(E^{\frac{2}{3}}) < S < O(E^{\frac{1}{2}})$ and $\sigma \geqslant O(1)$ to solve a steady flow problem. We now consider these parameter ranges and show how to exploit those boundary layers to complete the solution of the time-dependent spin-down problem. There were two primary purposes which motivated this exercise. First, it was desired to examine the specific manner in which the temperature boundary condition influences the solutions for very weak stratification, since as we shall note in the next section that condition is known to effect the spin-down process when the stratification is larger. Symptoms of this behaviour are indeed found in the stratification range assumed here. Secondly, it was planned to indicate how the aforementioned boundary-layer solutions require modification in order to usefully describe a reasonably representative time-dependent problem. The most distinctive feature occurs in the solution for the thickest boundary layer viewed as a function of Prandtl number.

For the sake of brevity no discussion will be given here of the origin and structure of the side-wall boundary layers; considerable insight may be gained from the clear treatment by Barcilon & Pedlosky (1967*b*). We introduce without further comment the boundary-layer variables

$$\xi = \frac{A-r}{E^{\frac{1}{4}}}\left(1+\lambda\frac{S}{E^{\frac{1}{2}}}+\ldots\right), \quad \eta = \frac{A-r}{S^{\frac{1}{2}}}, \quad \rho = \frac{A-r}{E^{\frac{1}{3}}S^{-\frac{1}{6}}}$$

for the 'diffusion', 'hydrostatic' and 'buoyancy' layers, respectively. The parameter λ is introduced because higher-order terms in the balance equations for the diffusion layer are required, and the constant λ might conceivably be necessary to keep the expansions uniformly valid in ξ. We denote by bar, caret and tilde the functions of boundary-layer form (i.e. which vanish for large values of the boundary-layer variable) with each of these regions as their domain.

We consider problem I as defined in the preceding section and scale the correction fields in the following way:

$$\begin{aligned}
\bar{v} &= \bar{v}^{(0)} + (S/E^{\frac{1}{2}})\bar{v}^{(1)} + \ldots,\\
\bar{\psi} &= E^{\frac{1}{2}}[\bar{\chi}^{(0)} + (S/E^{\frac{1}{2}})\bar{\chi}^{(1)} + \ldots],\\
\bar{p} &= E^{\frac{1}{4}}[\bar{p}^{(0)} + (S/E^{\frac{1}{2}})\bar{p}^{(1)} + \ldots],\\
\bar{T} &= SE^{-\frac{1}{4}}\bar{T}^{(0)} + \ldots.
\end{aligned}$$

$$\begin{aligned}
\hat{v} &= SE^{-\frac{1}{2}}\hat{v}^{(0)}, & \tilde{v} &= E^{\frac{1}{3}}S^{-\frac{1}{3}}\tilde{v}^{(0)},\\
\hat{\psi} &= E^{\frac{1}{2}}\hat{\chi}^{(0)}, & \tilde{\psi} &= E^{\frac{1}{2}}\tilde{\chi}^{(0)},\\
\hat{p} &= S^{\frac{3}{2}}E^{-\frac{1}{2}}\hat{p}^{(0)}, & \tilde{p} &= ES^{-\frac{1}{2}}\tilde{p}^{(0)},\\
\hat{T} &= S^{\frac{3}{2}}E^{-\frac{1}{2}}\hat{T}^{(0)}, & \tilde{T} &= S^{\frac{1}{2}}\tilde{T}^{(0)}.
\end{aligned}$$

From the boundary-layer approximations to (2.1) and (2.2), the governing equations may be written

$$\left.\begin{aligned}
-2\bar{v}^{(0)} &= \bar{p}^{(0)}_{\xi},\\
\bar{v}^{(0)}_{\tau} + 2\bar{\chi}^{(0)}_{z} &= \bar{v}^{(0)}_{\xi\xi},\\
\bar{p}^{(0)}_{z} &= 0,\\
\bar{T}^{(0)}_{\tau} + 4\bar{\chi}^{(0)}_{\xi} &= \sigma^{-1}\bar{T}^{(0)}_{\xi\xi}.
\end{aligned}\right\} \quad (3.3\,a\text{–}d)$$

$$\left.\begin{aligned}
-2\bar{v}^{(1)} &= \lambda\bar{p}^{(0)}_{\xi} + \bar{p}^{(1)}_{\xi},\\
\bar{v}^{(1)}_{\tau} + 2\bar{\chi}^{(1)}_{z} &= \bar{v}^{(1)}_{\xi\xi} + 2\lambda\bar{v}^{(0)}_{\xi\xi},\\
\bar{p}^{(1)}_{z} &= \bar{T}^{(0)}.
\end{aligned}\right\} \quad (3.4\,a\text{–}c)$$

$$\left.\begin{aligned}
-2\hat{v}^{(0)} &= \hat{p}^{(0)}_{\eta},\\
2\hat{\chi}^{(0)}_{z} &= \hat{v}^{(0)}_{\eta\eta},\\
\hat{p}^{(0)}_{z} &= \hat{T}^{(0)},\\
4\hat{\chi}^{(0)}_{\eta} &= \sigma^{-1}\hat{T}^{(0)}_{\eta\eta}.
\end{aligned}\right\} \quad (3.5\,a\text{–}d)$$

$$\left.\begin{aligned} -2\tilde{v}^{(0)} &= \tilde{p}^{(0)}_\rho, \\ 2\tilde{\chi}^{(0)}_z &= \tilde{v}^{(0)}_{\rho\rho}, \\ 0 &= \tilde{T}^{(0)}+\tilde{\chi}^{(0)}_{\rho\rho\rho}, \\ 4\tilde{\chi}^{(0)}_\rho &= \sigma^{-1}\tilde{T}^{(0)}_{\rho\rho}. \end{aligned}\right\} \quad (3.6\,a\text{–}d)$$

The only boundary conditions at the side wall which can and must be satisfied by the boundary-layer functions defined above are, from (2.5),

$$\left.\begin{aligned} V(r=A)+\bar{v}^{(0)}(\xi=0)+(S/E^{\frac{1}{2}})\,[\bar{v}^{(1)}(\xi=0)+\hat{v}^{(0)}(\eta=0)] &= 0, \\ \mathrm{X}(r=A)+\bar{\chi}^{(0)}(\xi=0)+\hat{\chi}^{(0)}(\eta=0)+\tilde{\chi}^{(0)}(\rho=0) &= 0, \\ \tilde{\chi}^{(0)}_\rho(\rho=0) &= 0, \\ \bar{T}^{(0)}(\xi=0) &= 0. \end{aligned}\right\} \quad (3.7\,a\text{–}d)$$

It is of course the necessity for satisfying these conditions, as well as the structure of the individual boundary layers, which precisely dictate our scaling of the boundary-layer variables. The relevant initial conditions are

$$\bar{v}^{(0)} = \bar{T}^{(0)} = \bar{v}^{(1)} = 0 \quad \text{on} \quad \tau = 0, \qquad (3.8$$

and although the compatibility condition (2.7) is itself only approximate, it may be applied to the boundary-layer functions here because: (1) the three side-wall layers are each thicker than the Ekman layer, and (2) the relevant higher-order terms in the side-wall layers are larger than the error in (2.7)

$$\bar{\chi}^{(0)} = \pm\tfrac{1}{2}\bar{v}^{(0)}, \quad \bar{\chi}^{(1)} = \pm\tfrac{1}{2}\bar{v}^{(1)}, \quad \hat{\chi}^{(0)} = 0 \quad \text{on} \quad z=\tfrac{1}{2}\pm\tfrac{1}{2}. \qquad (3.9\,a\text{–}c)$$

Equations (3.3)–(3.9) comprise the system we must solve, and again we shorten the details by omitting the more routine calculations. The diffusion layer has as its primary purpose the satisfaction of the no-slip boundary condition on the $O(1)$ azimuthal velocity. The lowest-order velocities, which are the same as those in a homogeneous fluid, are found from (3.3 *a–c*), (3.7 *a*), (3.8 *a*) and (3.9 *a*):

$$\left.\begin{aligned} \bar{v}^{(0)} &= -A\,e^{-2\tau}\,\mathrm{erfc}\,(\xi/2\tau^{\frac{1}{2}}), \\ \bar{\chi}^{(0)} &= -A(z-\tfrac{1}{2})\,e^{-2\tau}\,\mathrm{erfc}\,(\xi/2\tau^{\frac{1}{2}}), \\ \bar{w}^{(0)} = \bar{\chi}^{(0)}_\xi &= A\,e^{-2\tau}\,(\pi\tau)^{-\frac{1}{2}}\,(z-\tfrac{1}{2})\exp\,(-\xi^2/4\tau). \end{aligned}\right\} \quad (3.10\,a\text{–}c)$$

The lowest-order azimuthal velocity is independent of z, and the interior meridional circulation expressed by (3.2 *b*) is closed by the meridional flow in the diffusion layer

$$\mathrm{X}(r=A)+\bar{\chi}^{(0)}\,(\xi=0) = 0. \qquad (3.11)$$

This circulation, which exhibits the intimate coupling between the Ekman layer, interior, and diffusion layer, drives a temperature variation which is determined by (3.3 *d*), (3.7 *d*) and (3.8 *b*). For our purposes it is convenient to express $\bar{T}^{(0)}$ as a Laplace transform:

$$\begin{aligned} \bar{T}^{(0)}_L(\xi,z,s) &\equiv \int_0^\infty e^{-s\tau}\,\bar{T}^{(0)}\,(\xi,z,\tau)\,d\tau \\ &= 4A(z-\tfrac{1}{2})\,(s+2)^{-\frac{1}{2}}\,\sigma[s(1-\sigma)+2]^{-1}\,[e^{-\xi(s+2)^{\frac{1}{2}}}-e^{-\xi(s\sigma)^{\frac{1}{2}}}]. \end{aligned} \qquad (3.12)$$

As is obvious from (3.4), $\overline{T}^{(0)}$ generates a z-dependent (or 'baroclinic') portion of the higher-order azimuthal velocity in the diffusion layer. From (3.4 *a, c*) and (3.12), the expression for $\bar{v}_L^{(1)}$ may be conveniently written

$$\bar{v}_L^{(1)} = A\sigma(z-\tfrac{1}{2})^2[s(1-\sigma)+2]^{-1}\left[e^{-\xi(s+2)^{\frac{1}{2}}}-\left(\frac{s\sigma}{s+2}\right)^{\frac{1}{2}}e^{-\xi(s\sigma)^{\frac{1}{2}}}\right]+\alpha(\xi,s;\lambda), \quad (3.13)$$

the as yet undetermined function α representing the z-independent part of the higher-order azimuthal velocity. An equation for α is found by a procedure which is analogous to the derivation of the equation satisfied by $\bar{v}_L^{(0)}$, i.e. we integrate the Laplace transform of (3.4 *b*) with respect to z and apply (3.9 *b*) to yield

$$\alpha_{\xi\xi}-(s+2)\,\alpha = A\Big\{2\lambda\, e^{-\xi(s+2)^{\frac{1}{2}}}+\tfrac{1}{3}\sigma[s(1-\sigma)+2]^{-1}\,e^{-\xi(s+2)^{\frac{1}{2}}}$$
$$-\sigma[s(1-\sigma)+2]^{-1}\left(\frac{s\sigma}{s+2}\right)^{\frac{1}{2}}\left[\frac{1}{2}+\frac{s(1-\sigma)}{12}\right]e^{-\xi(s\sigma)^{\frac{1}{2}}}\Big\}.$$

The solution is given in terms of an arbitrary function $\beta(s)$ which will be determined from the boundary condition (3.7 *a*):

$$\alpha(\xi,s) = \{\beta(s)-A(s+2)^{-\frac{1}{2}}(\lambda+\tfrac{1}{6}\sigma[s(1-\sigma)+2]^{-1})\,\xi\}\,e^{-\xi(s+2)^{\frac{1}{2}}}$$
$$+A\sigma[s(1-\sigma)+2]^{-2}\left(\frac{s\sigma}{s+2}\right)^{\frac{1}{2}}\left[\frac{1}{2}+\frac{s(1-\sigma)}{12}\right]e^{-\xi(s\sigma)^{\frac{1}{2}}}.$$

The presence of a term multiplied by the boundary-layer variable ξ is a potential source of difficulty, since when $\xi = O[(S/E^{\frac{1}{2}})^{-1}]$ the second term in the perturbation series for $\bar{v}_L$ appears to be as important as the first term. To prevent the series from failing to be uniformly valid over the whole domain of ξ, therefore, it is necessary to choose $\lambda = -\frac{1}{12}$ when $\sigma = 1$. When $\sigma \neq 1$, however, the proper choice of λ is *zero*. We may see these results by comparing the asymptotic expansions for large ξ of $\bar{v}^{(0)}$ and the Laplace inversion of the offending portion of (3.13), namely

$$\gamma(s) \equiv -A(s+2)^{-\frac{1}{2}}(\lambda+\tfrac{1}{6}\sigma[(s+2)(1-\sigma)+2\sigma]^{-1})\,\xi\, e^{-\xi(s+2)^{\frac{1}{2}}}. \quad (3.14)$$

If we let $x = \xi 2^{-1}\tau^{-\frac{1}{2}}$, then for $\sigma = 1$ the inversion is

$$\gamma(\tau;\sigma=1) = -A\,e^{-2\tau}x(\lambda+\tfrac{1}{12})\,2\pi^{-\frac{1}{2}}e^{-x^2}. \quad (3.15)$$

Recalling the expansion

$$\operatorname{erfc} z \sim \pi^{-\frac{1}{2}}z^{-1}e^{-z^2}\{1+O(z^{-2})\} \quad (z\to\infty \text{ and } |\arg z| < 3\pi/4), \quad (3.16)$$

we find that the ratio of (3.15) to (3.10 *a*) is

$$\frac{x(\lambda+\frac{1}{12})\,2\pi^{-\frac{1}{2}}e^{-x^2}}{\pi^{-\frac{1}{2}}x^{-1}e^{-x^2}} = 2x^2(\lambda+\tfrac{1}{12})+O(1).$$

Unless $\lambda = -\frac{1}{12}$ this ratio is surely unbounded for large x. On the other hand, we may write the inversion of (3.14) for the case $\sigma < 1$ in the form

$$\gamma(\tau;\sigma<1) = -A\,e^{-2\tau}[x\lambda 2\pi^{-\frac{1}{2}}e^{-x^2}-\tfrac{1}{6}\tau^{\frac{1}{2}}\,\delta x\, e^{-\tau\delta^2}\operatorname{Im}\{e^{ix2\delta\tau^{\frac{1}{2}}}\operatorname{erfc}(x+i\delta\tau^{\frac{1}{2}})\}], \quad (3.17$$

where $\delta = [2\sigma/(1-\sigma)]^{\frac{1}{2}}$ and we have used (812.5) in Campbell & Foster (1948). Now from (3.16) we may approximate the function in the curly brackets of (3.17) by

$$e^{ix2\delta\tau^{\frac{1}{2}}}\operatorname{erfc}(x+i\delta\tau^{\frac{1}{2}}) \sim \frac{x-i\delta\tau^{\frac{1}{2}}}{\pi^{\frac{1}{2}}(x^2+\delta^2\tau)}e^{\delta^2\tau-x^2}\,(1+O[(x+i\delta\tau^{\frac{1}{2}})^{-2}]).$$

Then (3.17) becomes

$$\gamma(\tau;\sigma<1) \sim -A\,e^{-2\tau}[2\pi^{-\frac{1}{2}}\lambda x\,e^{-x^2}+\tfrac{1}{6}(\tau^{\frac{1}{2}}\delta)^2\,x\pi^{-\frac{1}{2}}(x^2+\delta^2\tau)^{-1}\,e^{-x^2}(1+O(x^{-2}))], \tag{3.18}$$

and the ratio of (3.18) to (3.10a) is

$$2x^2\lambda+O(1)+\frac{\delta^2\tau}{6}\frac{x^2}{x^2+\delta^2\tau}+O(x^{-2}).$$

The ratio is bounded for large x and finite $\delta^2\tau$ provided $\lambda = 0$. When $\sigma > 1$ the inversion of (3.14) is only slightly different from (3.17), and the conclusion $\lambda = 0$ is again necessary.

Before proceeding with the solutions in the other boundary layers, we should comment on the significance of the different values for λ. It is true, of course, that some sort of distinction between the mathematical representations of solutions for $\sigma = 1$ and for $\sigma \neq 1$ but $\sigma = O(1)$ is inherent to the detailed analysis of a large class of motions in a dissipative inhomogeneous fluid. It should be obvious that the behaviour of $\lambda(\sigma)$ here is a consequence of a non-uniformity of the limit $\sigma \to 1$. Since σ is the ratio of two physical parameters of the fluid, it appears justifiable to take the limit before computing a property of the solution (namely λ) rather than after, and this process leads without inconsistency to the results above. Because the difference in the values of λ amounts to a small shift in the effective boundary-layer thickness, even the lowest-order diffusion-layer solutions are affected. The basic question is, however, why should the limit $\sigma \to 1$ be non-uniform? The requirement of precisely identical diffusive properties of momentum and temperature imposes a kind of mathematical degeneracy on the flow equations. For our specific system of equations (2.1)–(2.3), the degeneracy is reflected in the fact that the governing equation for the stream function which may be derived when $\sigma = 1$ is sixth order in the spatial variables, instead of eighth order as when $\sigma \neq 1$. This situation differs from that considered by Barcilon & Pedlosky (1967b), who remarked that the parameters σ and S only appear multiplied together, indicating the absence of any special significance for $\sigma = 1$. Nevertheless, their diffusive boundary-layer solutions do require a correction factor λ for all values of σS. This is not at all surprising in view of the aforementioned degeneracy if we recall that their steady equations are reduced in order from the time-dependent equations (2.1)–(2.3) to the same extent as are the time-dependent equations when $\sigma = 1$.

With consistent expressions at hand for the Laplace transforms of the diffusion-layer flow, we can fulfill our goal of showing how the remaining boundary conditions are satisfied at the side wall. However, we should point out that complete inversion of the transforms $\bar{v}_L^{(1)}$ and $\bar{T}_L^{(0)}$ cannot be obtained in any reasonably illuminating form. We emphasize that for our purposes this is not a severe handicap because we are interested mainly in the roles and key properties of the boundary layers, which can be determined without finding the explicit τ-dependence of the solutions.

The form of the stream function in the buoyancy layer is found from (3.6),

$$\tilde{\chi}^{(0)} = e^{-\rho\sigma^{\frac{1}{4}}}[B(z,\tau)\sin(\rho\sigma^{\frac{1}{4}}+\tfrac{1}{4}\pi)+C(z,\tau)\cos(\rho\sigma^{\frac{1}{4}}+\tfrac{1}{4}\pi)] \quad (3.19)$$

and $C(z,\tau) = 0$ by (3.7 c) so that the largest vertical velocity component at the side wall vanishes. The solutions in the hydrostatic layer may be expressed in series using (3.5) and (3.9 c),

$$\left.\begin{aligned}\hat{\chi}^{(0)} &= \sum_{n=1}^{\infty} A^{(n)}(\tau)\frac{\sin n\pi z}{n\pi}e^{-\sigma^{-\frac{1}{2}}n\pi\eta},\\ \hat{v}^{(0)} &= 2\sigma\sum_{n=1}^{\infty} A^{(n)}(\tau)\frac{\cos n\pi z}{(n\pi)^2}e^{-\sigma^{-\frac{1}{2}}n\pi\eta}.\end{aligned}\right\} \quad (3.20)$$

Now the vanishing of the higher-order portion of boundary condition (3.7 a) requires that both the baroclinic and z-independent portions of $\bar{v}^{(1)}$ must be matched at the side wall, thereby determining all the $A^{(n)}$'s and the function $B(s)$ in $\alpha(\xi,s)$, cf. (3.13). The $O(E^{\frac{1}{2}})$ meridional circulation induced in the hydrostatic layer is closed by the buoyancy layer, which, as is now well known, acts dynamically much like the Ekman layer in requiring a mass influx for its maintenance. In this way the function B in (3.19) is fixed in terms of the known $A^{(n)}$'s by (3.7 b). We omit the details, noting only that we use the series expansion

$$(z-\tfrac{1}{2}) = -\sum_{n=1}^{\infty}[1+(-)^n]\frac{\sin n\pi z}{(n\pi)}$$

to arrive at

$$A_L^{(n)}(s) = -A[1+(-)^n][s(1-\sigma)+2]^{-1}\left[1-\left(\frac{s\sigma}{s+2}\right)^{\frac{1}{2}}\right],$$

$$\beta(s) = -A\sigma[s(1-\sigma)+2]^{-1}\left\{\frac{1}{12}\left[1-\left(\frac{s\sigma}{s+2}\right)^{\frac{1}{2}}\right] + [s(1-\sigma)+2]^{-1}\left(\frac{s\sigma}{s+2}\right)^{\frac{1}{2}}\left[\frac{1}{2}+\frac{s(1-\sigma)}{12}\right]\right\},$$

$$B_L(s,z) = -2^{\frac{1}{2}}A(z-\tfrac{1}{2})[s(1-\sigma)+2]^{-1}\left[1-\left(\frac{s\sigma}{s+2}\right)^{\frac{1}{2}}\right].$$

The inversions of B_L and $A_L^{(n)}$ are fairly complex except when $\sigma = 1$, when a simple expression is available from Campbell & Foster (1948),

$$\mathscr{L}^{-1}\left[1-\left(\frac{s}{s+2}\right)^{\frac{1}{2}}\right] = e^{-\tau}[I_0(\tau)-I(\tau)] \equiv \Delta(\tau),$$

$$\Delta(\tau\to 0) = 1, \quad \Delta(\tau\to\infty) \sim [2(2\pi)^{\frac{1}{2}}\tau^{\frac{3}{2}}]^{-1}.$$

The solution of problem II proceeds in much the same way, and in particular we find the same expressions for the lowest-order diffusive-layer velocity components and the same values for the factor λ. An important difference is that the baroclinic portion of $\bar{v}^{(1)}$ vanishes at the side wall, because (3.7 d) is replaced by the appropriate boundary condition for a wall with zero heat flux, $\overline{T}_\xi^{(0)}(\xi = 0) = 0$, which when combined with (3.4 a, c) implies $\bar{v}_z^{(1)}(\xi = 0) = 0$. This means that the agent which forces the presence of hydrostatic and buoyancy layers in problem I is absent in problem II, since in the latter case the z-independent function β is fully capable of satisfying the higher-order part of (3.7 *a*). Consequently, the hydrostatic and buoyancy layers disappear to the order to which we scaled them for problem I, i.e. they both contain meridional circulations smaller than the Ekman layer flux of $O(E^{\frac{1}{2}})$. Some detailed consideration reveals that the buoyancy layer must match the $O(E^{\frac{1}{4}})$ vertical velocity of the diffusion layer at the wall, thereby specifying the strength of the buoyancy layer flux as $O(E^{\frac{3}{4}}S^{-\frac{1}{4}})$. The complete determination of all the arbitrary parameters of the hydrostatic and buoyancy layers is rather complicated, because the relevant boundary conditions become tangled up with higher-order terms from the diffusion layer.

In summary, we remark first that the stratification does not inhibit the complete removal of the initially imposed $O(1)$ vorticity during the time scale $\tau = O(1)$ either in the interior or throughout the boundary layers. The most significant effects of the stratification are the shift in the diffusive boundary-layer thickness when $\sigma = 1$ and the z-dependence of the velocity component $\bar{v}^{(1)}$ throughout the diffusion layer. Whether the higher-order baroclinic azimuthal velocity is dynamically important depends on whether it is capable of driving the hydrostatic and buoyancy layers, which in turn depends on the temperature boundary condition as we have seen. The buoyancy-layer flux for problem I remains $O(E^{\frac{1}{2}})$ as S increases from $O(E^{\frac{2}{3}})$ to $O(E^{\frac{1}{2}})$, but for problem II it decreases from $O(E^{\frac{7}{12}})$ to $O(E^{\frac{5}{8}})$. Now the significance of this extremely small difference in circulation strength is that it is a consequence of the mechanism by which the thermal boundary condition weakens the circulation in the particular boundary layer which is most characteristic of a stratified fluid. The same mechanism causes the interior flow properties to differ with the thermal boundary condition when the stratification is much larger.

We conclude this section with some brief remarks on the form of the solutions for values of S outside but nearby the range $O(E^{\frac{2}{3}}) < S < O(E^{\frac{1}{2}})$. For smaller S, the two Stewartson layers exist along the side wall. It can be shown that the lowest-order solutions in the layers with stratification are the same as those for a homogeneous fluid when $S < O(E^{\frac{3}{4}})$ for problem I and when $S < O(E^{\frac{2}{3}})$ for problem II. Once again the significant effect is whether the largest baroclinic azimuthal velocity component in the outer layer is z-dependent at the wall. For problem I the flux in the inner Stewartson layer increases like $O(SE^{-\frac{1}{6}})$ for $S \geqslant O(E^{\frac{3}{4}})$, which is consistent with both the homogeneous value $O(E^{\frac{7}{12}})$ (Greenspan & Howard 1963) and the $O(E^{\frac{1}{2}})$ flux at the $O(E^{\frac{2}{3}})$ stratification where the buoyancy layer first appears. When $S = O(E^{\frac{1}{2}})$ the hydrostatic layer has thickened to the extent that it has merged with the diffusion layer, a complication tending to confuse the simpler picture of the roles of the individual layers for smaller S. However, on the basis of that picture, the following easily demonstrable results should appear at least plausible, if not obvious. For problem II the combined $O(E^{\frac{1}{4}})$ layer returns the $O(E^{\frac{1}{2}})$ flux from the interior to the Ekman layers and has a baroclinic lowest-order azimuthal velocity. The buoyancy layer then matches the largest vertical velocity component at the side wall and requires an $O(E^{\frac{5}{8}})$ flux. In problem I the combined layer has a z-independent lowest-order azimuthal velocity and only the buoyancy layer possesses an $O(E^{\frac{1}{2}})$ meridional circulation with which to close the interior secondary flow. In this case an $O(E^{\frac{1}{4}}) \times O(E^{\frac{1}{4}})$ corner region is necessary at the top and bottom of the combined layer in order to satisfy no slip conditions on the z-independent azimuthal velocity. The main reason for mentioning here the results for $S \leqslant O(E^{\frac{2}{3}})$ and $S = O(E^{\frac{1}{2}})$ is to lend more credence to those for intermediate values of S by suggesting the similarities near the points of common applicability. The details of the former are omitted because they are lengthy and are not intrinsically interesting.

4. Potential vorticity conservation

It follows immediately from (2.1) to (2.3) that

$$\frac{\partial}{\partial t}[2\hat{\boldsymbol{k}}\,.\,\nabla \times \mathbf{u} + S^{-1}\hat{\boldsymbol{k}}\,.\,\nabla T] = O(E, E\sigma^{-1}). \tag{4.1}$$

When S is order one the balance between changes in relative vorticity and the horizontal deflexion of the constant temperature lines controls motions in the interior region, for time scales shorter than those of viscous dissipation and heat conduction (in fact, it may be appropriate to define the extent of the interior as the region where this balance holds). Because of the conservation of the quantity in (4.1), the potential vorticity, the solution of the spin-down problem on the τ time scale represents only a rearrangement of the potential vorticity which is imposed by the initial change in rotation rate. The redistribution is accomplished by the thinnest boundary layers, which modify the interior flow in such a way that the motions remaining when τ becomes large satisfy certain boundary conditions, specifically those for which the thinnest layers themselves are required at finite values of τ.

The essential linear dynamics governing spin down in a rotating stratified fluid are thus quite simple, a fact worth emphasizing in view of the analytic complications which arose in several earlier treatments. Since a correct solution for the transient development of the potential vorticity in a cylindrical container was first published by Walin (1969), our aim in this section is to explore some aspects of the spin-down process with $S = O(1)$ which have not yet been completely resolved.

Although the thermal boundary condition is known to influence the interior flow, the characteristics of the effects have not been fully described, and a rather special solution for a spherical container is quite useful for this purpose. The solution illustrates the most effective procedure for finding expressions for all or part of the flow and, in addition, demonstrates a type of boundary-layer coupling which cannot occur in a cylinder but which is representative of all non-cylindrical geometries. Results of Pedlosky (1967) and Sakurai (1969a, b) for the cylinder are also clarified. A comparison between predictions of the linear model and laboratory observations of spin down by Holton (1965) indicates that even though there is qualitative agreement between the proper theoretical solution and data, the agreement is not nearly so good as comparisons in Holton's paper suggest. We conclude with a comment on the eventual dissipation of the imposed potential vorticity in the interior region.

Interest in the spin-down problem has been confined to rotating stratified fluids in cylindrical containers, largely because any other container shape aggravates the already severe experimental difficulty of producing a stable state of stratification. None the less, even with this simplest imaginable geometry, the nature of the boundary conditions for (4.1) caused theoretical difficulties in the analytical description of spin down. Although the asymptotic solution for large τ is known, as yet the time-dependent solution for problem I has not been obtained, and consequently the manner in which the interior flow pattern evolves is uncertain. In order to improve understanding of the time development for fixed temperature boundary conditions, it was felt that a theoretical examination of a fluid contained in a sphere would be potentially helpful. The simplest procedure to obtain a solution is to first find the interior asymptotic azimuthal velocity and temperature fields, which by conserving potential vorticity satisfy the following equation in cylindrical co-ordinates,

$$2r^{-1}(rV_{\infty})_r + S^{-1}T_{\infty z} = 4, \tag{4.2}$$

along with a homogeneous boundary condition on the sphere $r^2+z^2 = \rho^2 = 1$ (the sphere radius is the length scale H). This solution by itself obviously may not satisfy the spin-down problem. However, the initial-value problem, and any other one compatible with the quasi-geostrophic model for that matter, can be solved providing a spatially complete set of solutions can be found to the homogeneous form of (4.1) with a homogeneous but time-dependent boundary condition. These solutions are the eigenfunctions in a problem for which the eigenvalues are the modal decay rates. The known solutions are then superposed in such a way that in combination with the asymptotic solution the initial conditions are satisfied. We shall demonstrate that it is easy to find the asymptotic solutions for the spin-down problem and that separable time-dependent solutions are accessible only in the special circumstance when the product σS is exactly one. In that case only one time-dependent mode is required to satisfy spin-down initial conditions.

Determination of the boundary conditions for the interior problem of course requires analysis of the boundary layers, which may be represented by the variables

$$\zeta = \frac{1-\rho}{E^{\frac{1}{2}}}, \quad \xi = \frac{1-\rho}{E^{\frac{1}{4}}}.$$

These layers are referred to here as the 'inner' and 'outer' or 'diffusion' layers respectively, and to emphasize their relation to those in the cylinder, we denote by tilde and bar the boundary-layer correction functions corresponding to each of these regions. We use spherical co-ordinates (ρ, θ, ϕ) and scale the correction fields in the following manner:

$$\begin{aligned} \tilde{\mathbf{U}}\cdot\hat{\boldsymbol{\rho}} &= \tilde{\alpha} = E^{\frac{1}{2}}\tilde{\alpha}^{(0)}, & \bar{\alpha} &= E^{\frac{3}{4}}\bar{\alpha}^{(0)}, \\ \tilde{\mathbf{U}}\cdot\hat{\boldsymbol{\theta}} &= \tilde{\beta} = \tilde{\beta}^{(0)}, & \bar{\beta} &= E^{\frac{1}{2}}\bar{\beta}^{(0)}, \\ \tilde{\mathbf{U}}\cdot\hat{\boldsymbol{\phi}} &= \tilde{v} = \tilde{v}^{(0)}, & \bar{v} &= \bar{v}^{(0)}, \\ \tilde{T} &= \tilde{T}^{(0)}, & \bar{T} &= \bar{T}^{(0)}, \\ \tilde{p} &= E^{\frac{1}{2}}\tilde{p}^{(0)}, & \bar{p} &= E^{\frac{1}{4}}\bar{p}^{(1)}. \end{aligned}$$

The boundary-layer approximations to (2.1)–(2.3) are written next, noting that (2.1) has three components and that $\hat{k} = \cos\theta\hat{\rho} - \sin\theta\hat{\theta}$:

$$\left.\begin{aligned} 2\hat{\boldsymbol{\rho}}.\hat{\boldsymbol{k}}\times\tilde{v}^{(0)}\hat{\boldsymbol{\phi}} - \hat{\boldsymbol{\rho}}.\hat{\boldsymbol{k}}\tilde{T}^{(0)} - \tilde{p}^{(0)}_{\zeta} &= 0,\\ 2\hat{\boldsymbol{\theta}}.\hat{\boldsymbol{k}}\times\tilde{v}^{(0)}\hat{\boldsymbol{\phi}} - \hat{\boldsymbol{\theta}}.\hat{\boldsymbol{k}}\tilde{T}^{(0)} &= \tilde{\beta}^{(0)}_{\zeta\zeta},\\ 2\hat{\boldsymbol{\phi}}.\hat{\boldsymbol{k}}\times\tilde{\beta}^{(0)}\hat{\boldsymbol{\theta}} &= \tilde{v}^{(0)}_{\zeta\zeta}\\ 4S\hat{\boldsymbol{k}}.\hat{\boldsymbol{\theta}}\tilde{\beta}^{(0)} &= \sigma^{-1}\tilde{T}^{(0)}_{\zeta\zeta}\\ \hat{\boldsymbol{\rho}}.\nabla\times(\hat{\boldsymbol{\rho}}\times\tilde{\beta}^{(0)}\hat{\boldsymbol{\theta}}) - \tilde{\alpha}^{(0)}_{\zeta} &= 0. \end{aligned}\right\} \quad (4.3\,a\text{–}e)$$

$$\left.\begin{aligned} 2\hat{\boldsymbol{\rho}}.\hat{\boldsymbol{k}}\times\bar{v}^{(0)}\hat{\boldsymbol{\phi}} - \hat{\boldsymbol{\rho}}.\hat{\boldsymbol{k}}\bar{T}^{(0)} - \bar{p}^{(1)}_{\xi} &= 0,\\ 2\hat{\boldsymbol{\theta}}.\hat{\boldsymbol{k}}\times\bar{v}^{(0)}\hat{\boldsymbol{\phi}} - \hat{\boldsymbol{\theta}}.\hat{\boldsymbol{k}}\bar{T}^{(0)} &= 0,\\ \bar{v}^{(0)}_{\tau} + 2\hat{\boldsymbol{\phi}}.\hat{\boldsymbol{k}}\times\bar{\beta}^{(0)}\hat{\boldsymbol{\theta}} &= \bar{v}^{(0)}_{\xi\xi},\\ \bar{T}^{(0)}_{\tau} + 4S\hat{\boldsymbol{k}}.\hat{\boldsymbol{\theta}}\bar{\beta}^{(0)} &= \sigma^{-1}\bar{T}^{(0)}_{\xi\xi},\\ \hat{\boldsymbol{\rho}}.\nabla\times(\hat{\boldsymbol{\rho}}\times\bar{\beta}^{(0)}\hat{\boldsymbol{\theta}}) - \bar{\alpha}^{(0)}_{\xi} &= 0. \end{aligned}\right\} \quad (4.4\,a\text{–}e)$$

The scaling of the dependent variables is not a necessary step, but here as in § 3 it serves to greatly simplify the presentation of the boundary-layer equations.

The boundary condition satisfied by the interior fields as τ becomes large is required first. Now the inner layer demands an $O(E^{\frac{1}{2}})$ ratio between the magnitude of the velocity normal to the boundary and the magnitude of the temperature and tangential velocity fields, as is clear from (4.3) and may be anticipated from the connexion between the dynamics of this layer and the operation of the buoyancy and Ekman layers. Consideration of (2.1)–(2.3) demonstrates that on time scales longer than $O(E^{-\frac{1}{2}})$ the interior region cannot support radial velocities as large as $O(E^{\frac{1}{2}})$, and consequently the inner layer cannot maintain order one temperature and tangential velocity fields on such time scales. Only the outer layer, thickened by diffusion beyond a distance $O(E^{\frac{1}{2}})$ from the boundary, is therefore capable of sustaining motions for $\tau\to\infty$ as large as the interior azimuthal velocity and temperature fields. Furthermore, the fields in the outer layer are connected by (4.4 *b*), since fluid particles undergoing azimuthal motion are deflected by the Coriolis force across the isodensity lines of the mean stratification, a situation which can be sustained only if the particle density adjusts itself to exactly match the density of its new surroundings. The combination of (4.4 *b*) and the boundary conditions

$$V_{\infty}(\rho = 1) + \bar{v}^{(0)}(\xi = 0) = 0,$$
$$T_{\infty}(\rho = 1) + \bar{T}^{(0)}(\xi = 0) = 0,$$

insists that the interior fields on their own satisfy

$$2\cos\theta V_{\infty} - \sin\theta T_{\infty} = 0 \quad \text{on} \quad \rho = 1. \tag{4.5}$$

The solutions to (4.2) and (4.5) which satisfy the additional obvious requirements of geostrophic and hydrostatic balance (3.1 *a*, *c*) are

$$V_{\infty} = \frac{2S}{2S+1}r, \quad T_{\infty} = \frac{2S}{2S+1}2z. \tag{4.6}$$

For a nearly homogeneous rotating fluid spin down is essentially completed as $\tau \to \infty$, while for a strongly stratified rotating fluid the initial azimuthal velocity is only slightly modified and an increase in the stratification arises to ensure potential vorticity conservation. The particular simplicity of the solution (4.6) results from the constant initial distribution of potential vorticity, although (4.5) must apply for any initial conditions.

We next consider the determination of the time-dependent solutions on the time scale $\tau = O(1)$. The boundary conditions to be satisfied are

$$\left.\begin{aligned} T(\rho = 1) + \bar{T}^{(0)}(\xi = 0) + \tilde{T}^{(0)}(\zeta = 0) &= 0, \\ V(\rho = 1) + \bar{v}^{(0)}(\xi = 0) + \tilde{v}^{(0)}(\zeta = 0) &= 0, \\ \tilde{\beta}^{(0)}(\zeta = 0) &= 0, \\ \hat{\boldsymbol{\rho}} . \mathbf{U}(\rho = 1) + E^{\frac{1}{2}}\tilde{\alpha}^{(0)}(\zeta = 0) &= 0. \end{aligned}\right\} \quad (4.7\,a\text{–}d)$$

Using (4.4 b) and the analogous relation derived from (4.3 c) and (4.3 d) for functions in the inner layer,

$$[2\sigma S \sin\theta \tilde{v}^{(0)} + \cos\theta \tilde{T}^{(0)}]_{\zeta\zeta} = 0,$$

(4.7 a) and (4.7 b) may be combined to yield the single condition

$$\begin{aligned} 2\tilde{v}^{(0)}(\zeta = 0)\,[\sigma S \sin^2\theta + \cos^2\theta] \\ = \cos\theta\,[\sin\theta T(\rho = 1) - 2\cos\theta V(\rho = 1)]. \end{aligned} \quad (4.8)$$

From (4.3 b–d) the governing equation in the inner layer in terms of a single variable is

$$\tilde{v}^{(0)}_{\zeta\zeta\zeta\zeta\zeta\zeta} + 4\tilde{v}^{(0)}_{\zeta\zeta} m^4 = 0, \quad m^4 = \sigma S \sin^2\theta + \cos^2\theta,$$

and the boundary-layer solution satisfying (4.7 c) and (4.8) is

$$\begin{aligned} \tilde{v}^{(0)} &= (\cos\theta/2m^4)\,[\sin\theta T(\rho = 1) - 2\cos\theta V(\rho = 1)]\, e^{-m\zeta}\cos m\zeta, \\ \tilde{\beta}^{(0)} &= (1/2m^2)\,[\sin\theta T(\rho = 1) - 2\cos\theta V(\rho = 1)]\, e^{-m\zeta}\sin m\zeta. \end{aligned}$$

Then (4.3 e) and (4.7 d) give the boundary condition on the interior circulation,

$$\hat{\boldsymbol{\rho}} . \mathbf{U}(\rho = 1) = E^{\frac{1}{2}} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left\{ \frac{\sin\theta}{4m^3} [\sin\theta T(\rho = 1) - 2\cos\theta V(\rho = 1)] \right\}. \quad (4.9)$$

The solution in the outer layer, satisfying (4.7 b) and the equation

$$\bar{v}^{(0)}_{\tau} - \sigma m^{-4}[S\sin^2\theta + \cos^2\theta]\,\bar{v}^{(0)}_{\xi\xi} = 0,$$

is not needed explicitly to solve the interior problem, although for the time-dependent solution here as for the asymptotic solution before, a knowledge of the role of the outer layer in satisfying the boundary conditions is essential.

The time-dependent solutions are obtained by solving (4.1) and (4.9), written with the help of (3.1 b) and (3.1 d) in terms of the stream function defined in § 3:

$$\left[\frac{1}{r}(r\psi)_r\right]_r + S^{-1}\psi_{zz} = 0,$$

$$\left[\frac{z}{r}\frac{\partial}{\partial r}r - r\frac{\partial}{\partial z}\right]\{r\psi_r + S^{-1}z\psi_z + \psi + m^3 S^{-1}\psi_{\tau}\} = 0 \quad \text{on} \quad r^2 + z^2 = 1.$$

Separable solutions of this system exist when $m = 1$, and since we regard the Prandtl number as representing a fixed property of the fluid, the requirement on m means that the solutions are restricted to the particular stratification with $S = \sigma^{-1}$. We note that when $\sigma S = 1$ the buoyancy and Ekman layers in the cylinder have exactly the same thickness, so in spherical geometry the constraint implies that buoyant and Coriolis effects are exactly comparable in the inner boundary layer. Although this severe parametric restriction is indispensable in order to determine mathematical expressions, there is no pressing physical reason why the solutions should differ significantly if σS were close to but not precisely one.

When $S \geqslant 1$ the solutions separate in prolate spheroidal co-ordinates (ξ, u) defined by

$$r = (1-u^2)^{\frac{1}{2}}(\xi^2-(S-1))^{\frac{1}{2}} \quad \text{and} \quad z = S^{\frac{1}{2}}\xi u$$

and they may be expressed in terms of associated Legendre polynomials

$$\left.\begin{aligned} \psi(\xi,u,\tau) &= P_n^1\left(\frac{\xi}{(S-1)^{\frac{1}{2}}}\right)P_n^1(u)\,e^{-\lambda\tau} \quad (n \geqslant 1),\\ \lambda &= nS\frac{\alpha P_n^1(\alpha)-P_{n+1}^1(\alpha)}{\alpha P_n^1(\alpha)} \quad \left[\alpha \equiv \left(\frac{S}{S-1}\right)^{\frac{1}{2}}\right],\\ \lambda &\to -(n+1) \quad (S \to 1). \end{aligned}\right\} \tag{4.10}$$

The extension to the case $S \leqslant 1$ using oblate spheroidal co-ordinates is straightforward, but it is worth mentioning that only for modes $n = 1$ and 2 the spatial structure is independent of S and the corresponding decay rates are 2 and $2S+1$, respectively. The first mode represents a uniform vertical flow with everywhere constant temperature and zero azimuthal velocity, while the second mode has for its streamlines a family of hyperbolas in the r, z plane with asymptotic lines $r = 0$ and $z = 0$. This mode, in addition, has an everywhere uniform vertical component of vorticity and is thus the only mode excited in spin down. Using (4.6) the especially simple solution to the spin-down problem in the sphere when $\sigma S = 1$ is therefore

$$V = \frac{r}{2S+1}(2S+e^{-(2S+1)\tau}), \quad T = \frac{2z}{2S+1}2S\,(1-e^{-(2S+1)\tau}). \tag{4.11}$$

From (4.11) it is clear that the vorticity relative to the rotating reference frame decays uniformly inside the sphere, in sharp contrast to the result for a homogeneous fluid (Greenspan 1968, p. 63) for which the relative vorticity decays more rapidly near the equator and actually changes sign as τ increases. The small S limit of the expression for the vorticity derived from (4.11) is in fact identical to the homogeneous fluid solution near the poles of the sphere, and the qualitative disagreement as the equator is approached is due to the increasing dominance of buoyancy effects in the inner boundary layer implicit in (4.11). This dominance retards the development of the interior vertical velocity which is necessary for the effective operation of the vortex tube stretching mechanism. In this way the small S limit of (4.11) describes the behaviour of a fluid in which the only *direct* influence of the stratification occurs on the structure of the dissipative layers on non-horizontal boundaries. If in this limit the effects of stratification in the dissipative layers are suppressed by completely preventing thermal diffusion or by imposing thermal insulation on the boundary, then the homogeneous result will be applicable in the interior. On the other hand, if $S = O(1)$ with either of these conditions, then the initial order one azimuthal velocity will not vary at all on the τ scale, since a strong inner boundary layer cannot form and no condition analogous to (4.5) holds.

The difference in flow patterns depending on the thermal boundary condition is of course one of the issues confronted for weak stratification in § 3. In fact the existence of the modal solutions (4.10) requires the boundary conditions (2.5) instead of (2.6), and these modes rearrange the initial potential vorticity so that (4.5) can be satisfied after the inner layer decreases in strength. In a cylindrical container with thermally insulated conditions (2.6), the potential vorticity is redistributed by the Ekman layers so that the interior motion remaining when τ becomes large satisfies an analogy to (4.5), i.e. the no-slip condition on the azimuthal velocity on the horizontal boundaries. With conditions (2.5) both Ekman and buoyancy layers adjust the potential vorticity because the asymptotic interior motions must satisfy the temperature boundary condition at the side wall as well as the no-slip requirement on the horizontal surfaces. There is a clear distinction in the interior meridional flows during the τ time scale in each case. The insulation condition, demanding that the total normal temperature gradient vanish at the side wall, restricts most severely the gradient in the thinnest (buoyancy) layer, or in other words inhibits heat exchange between fluid and wall as fluid elements travel vertically. This amounts to a suppression of buoyancy-layer motions comparable to those in the Ekman layer, and consequently the entire meridional circulation tends to be confined near the Ekman layers. The distance of significant penetration of the interior motions from a horizontal boundary was found to be $O(S^{-\frac{1}{2}})$ by Walin (1969) and earlier by Lineykin (1955). Both Pedlosky (1967, equation (6.30)) and Sakurai (1969*a*, equation (63)) indicated that with fixed temperature conditions there are modal solutions in which significant motions are not confined in this manner. This is not unexpected if we apply the understanding gained from the modes (4.10) in a sphere and regard the solutions in the cylinder as necessary to redistribute potential vorticity so that the side-wall temperature boundary condition can be satisfied for large τ. One rather surprising consequence of this interpretation is that Pedlosky's solution (6.30), which is convincing by its very simplicity, cannot be the only time-dependent mode excited in the spin-down problem even for the particular parameter relation given by Pedlosky. By itself this mode conserves potential vorticity, and there is no way to add to it a time-independent part which takes up the initial potential vorticity, ensures that the sum will satisfy both initial conditions, and allows the steady portion to satisfy the necessary side-wall boundary condition. The time-dependent portion of the azimuthal velocity in spin down evidently must vary with z, and for all values of σ and S the modal sum in a cylinder with constant temperature boundary conditions is complicated.

The work summarized by table 1 in Sakurai (1969*b*) indicates the quantitative effect of the side-wall temperature boundary condition on the total angular momentum remaining in the interior of a cylinder as $\tau \to \infty$. Such results may be obtained much more directly by simply solving the equation of conservation of potential vorticity with the appropriate asymptotic (large τ) boundary conditions, thereby avoiding the involved computations for the time-dependent solutions. Furthermore, although the asymptotic total interior angular momentum is a reasonable measure of spin-down *per se*, the ease of finding the asymptotic solutions permits pointwise determination of the flow quantities. For example, since $2\hat{\mathbf{k}}.\nabla\times\mathbf{u}+S^{-1}T_z = 4$, contours for large τ of constant vertical vorticity show the spatial distribution of the conversion of the initial vorticity. The basic point is that potential vorticity conservation is surely the most potent means of describing the dynamics of nearly geostrophic motions in a rotating stratified fluid, and it should be exploited fully whenever it is applicable. Ignoring its implications would lead to confusion from the following sentence in Sakurai (1969*b*, §5): "The case with infinitely large radius is equivalent to that with vanishing effect of stratification." This is of course true in so far as the attainment of rigid rotation is concerned, but for a cylinder of infinitely large radius with $S \neq 0$, a temperature gradient constant in z must develop as $\tau \to \infty$ to conserve potential vorticity. This gradient is obviously absent in the homogeneous case studied by Greenspan & Howard (1963), and it points up the dynamical differences in the cases $r_0 = A \to \infty$ and $S \to 0$. A further cautionary note is that the limit $r_0 \to \infty$ in Sakurai's solutions (70) and (71) is non-uniform, as is particularly clear from (71); weakly converging series have been a constant source of errors in solutions of spin down. Another formulation for the infinite disk problem which emphasizes potential vorticity conservation was given by Walin (1969).

Holton (1965) found a solution to (4.1) for a spin-up problem (with homogeneous initial conditions and inhomogeneous Ekman boundary conditions), and in addition he made certain measurements of the angular velocity remaining as $\tau \to \infty$. It is now known that the time-dependent portion of his solution is incorrect and that the time-independent constituent corresponds to problem I boundary conditions. However, Holton stratified his fluid with salt, so that the conditions of problem II and hence Walin's (1969) solution (5.23) are applicable to his experiments. Making some minor modifications of Walin's expression for large τ and performing the series summations by standard means, we can directly compare the data presented by Holton with the correct asymptotic solution, shown by the solid curves in figures 1 and 2 (the boxes indicate the accuracy to which Holton's data was read from his figures). The dotted curves are those which Holton compared with his data, and in addition to being qualitatively very similar to the solid curves, their fit to the data is remarkably accurate. The only presently available experimental data for stratified spin down are therefore not in particularly good agreement with the linear theory formulated in §2. Holton's data shows less inhibition of the attainment of rigid rotation than would be expected for a stratified fluid on the basis of Walin's solution. The most probable causes are the slight motions in the basic state caused by curvature of the mean isodensity lines (Barcilon & Pedlosky 1967*a*; Phillips 1970) and the instabilities prevalent near container boundaries, both of which processes tending to reduce the effective stratification.

Since both solid curves are based on Walin's results, it is appropriate at this point to make two comments connected with their applicability. An experimental and theoretical estimate is sorely needed for the non-dimensional length scale l_c (see Walin 1969, p. 305) of the region where the initial conditions on the τ time scale are smoothed to zero, since only the crude bound $l_c > O(E^{\frac{1}{2}})$ is now certain. This quantity should be distinguished from the length l of the region starting at the

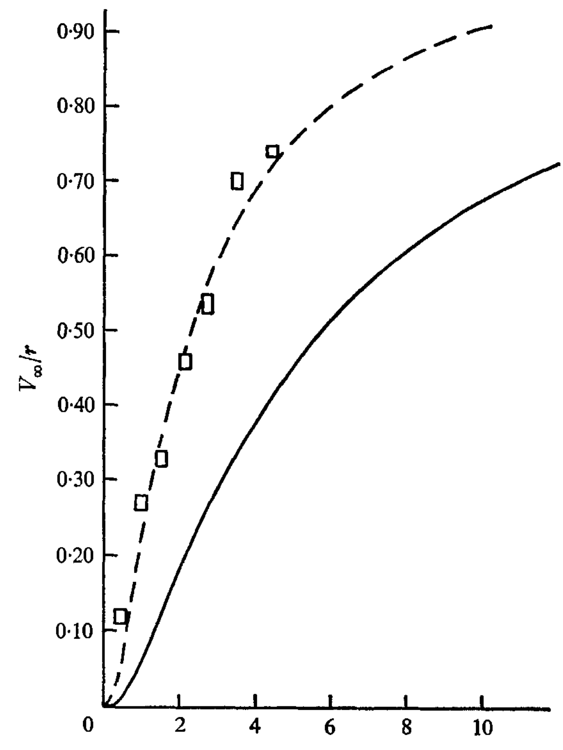

FIGURE 1. Angular velocity for $\tau \to \infty$ at $r = \frac{1}{2}A$ and the midplane of the cylinder, plotted against S^{-1}. ----, from Holton (1965); ——, from Walin (1969); □, from Holton (1965).

corner where the flux leaves the boundary layer. The vertical velocity at the edge of the boundary layer associated with Walin's solution (5.23) can be shown to vanish for a point $R < A$ for any $\tau > 0$, so that $l = O(1)$ because the Ekman layer flux is not unidirectional along the horizontal extent of the layer. It is not necessarily true that l_c is $O(1)$, however. In addition, Walin suggests that the condition $\epsilon \ll E^{\frac{1}{2}}$ may be replaced by $\epsilon \ll 1$ for axisymmetric problems like spin down. More precisely, the spin-down dynamics described in his paper and in this one require that ϵ should be small compared to one and that derivatives with respect to ϕ of the lowest-order azimuthal velocity and temperature fields should be $O(E^{\frac{1}{2}})$ smaller than the fields themselves. These constraints are weaker than the strict axisymmetry and bound on ϵ imposed formally in §2 and are more likely to be satisfied under experimental conditions.

The essential consequence of spin down governed by the constraint of potential vorticity conservation is that the adjustment of the fluid to a state of rigid rotation is almost never achieved until that constraint is broken (Sakurai's similarity solution for two infinite disks is the exception). If the very stringent assumption is made that equations (2.1)–(2.3) hold throughout the fluid during the time scales when the potential vorticity is dissipated, and if the stratification is strong enough and the thermal diffusion not so rapid that potential vorticity is

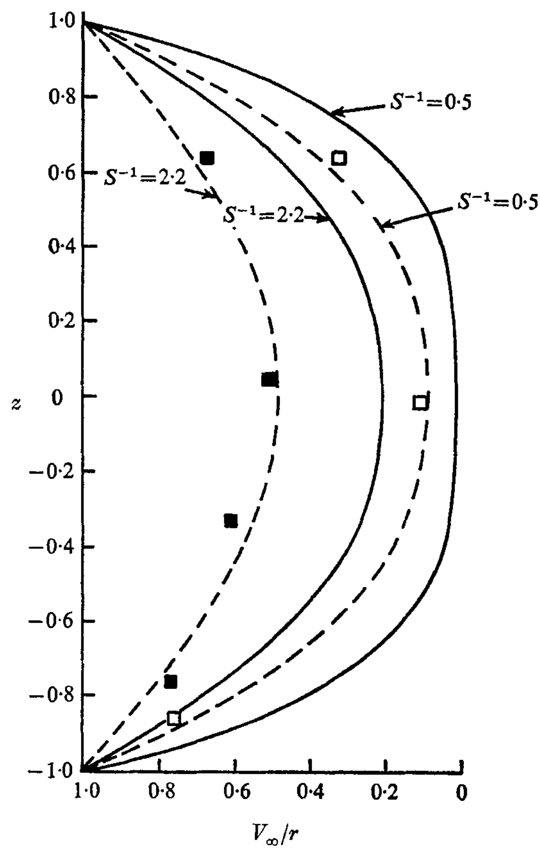

FIGURE 2. Angular velocity for $\tau \to \infty$ at $r = \frac{1}{2}A$, plotted against z for $S^{-1} = 2{\cdot}2$ and $S^{-1} = 0{\cdot}5$. – – – –, from Holton (1965); ——, from Walin (1969); ■, □, from Holton (1965).

lost before the τ time scale, then the complete decay of the initially imposed vorticity occurs on the shorter of the two time scales E^{-1} and (σ/E). This conclusion can be demonstrated by careful although somewhat lengthy analysis, but because of the arbitrary nature of the assumption of the validity of (2.1)–(2.3), it is sufficient here to just indicate that the conclusion rests on two basic considerations. First, the initial conditions for determination of the dynamics on any longer time scale may be taken as the asymptotic solutions remaining as $\tau \to \infty$. This is of course the same principle used in solving the problem on the τ time scale, and doing so on the longer time scales avoids inconsistencies of the type with which Holton & Stone (1968) were concerned. Secondly, considering for the moment $\sigma \gg 1$, it is easy to see that no order one z-dependent azimuthal velocity is permitted on the longer time scale σ/E. For from the azimuthal component of (2.1), if $v = O(1)$ than $u = O(E)$, and (2.3) requires $w = O(E)$ also. The thermal wind relation from the r and z components of (2.1) means that if $v_z = O(1)$, then $T_r = O(1)$. But then (2.2) implies w is zero to order E, which says that in fact $v_z = 0$ to order one. Since no boundary layers occur on the long time scale, the no-slip condition requires $v = 0$. When $\sigma \to 0$ a similar argument shows that no order one azimuthal velocity is allowed on the longer time E^{-1}. These two results are tantamount to determining the decay time of the initial vorticity.

Interest in the eventual fate of the potential vorticity in stratified fluids with varying Prandtl numbers is motivated by two illustrations of spin down in the recent literature, the solar application with small Prandtl number and stratification by salt with large Prandtl number. Howard (unpublished) described the very different dynamics in the limiting case $\sigma = O(E^{\frac{1}{2}})$ and verified complete spin down in the τ time scale, extending to and quantifying for a particular time-dependent flow the observation of Greenspan (1968, p. 126) that the steady flow when diffusion overwhelms conduction is in overall effect related to that in a homogeneous fluid. In an experiment designed to provide precision velocity measurements, McDonald & Dicke (1967) demonstrated that spin up in a cylinder stratified by cupric nitrate (S approximately 3 and σ large) was well described by a version of the solution of Pedlosky (1967) based on the diffusive dissipation of potential vorticity and, in particular, was completed within the time predicted by neglecting density diffusion. It should be emphasized that unless the parameters ϵ and F_R satisfy more severe restrictions than those specified in § 2, processes connected with non-infinitesimal perturbation amplitude or with non-negligible centrifugal bending of the mean isodensity lines violate potential vorticity conservation earlier than diffusive effects acting alone in the interior. When and how this occurs is still uncertain.

It is a pleasure to acknowledge conversations with L. N. Howard and J. Pedlosky on related topics which comprised theses submitted to the Department of Mathematics, M.I.T. (May 1967 and August 1968). This work was partially supported by the National Science Foundation through Grant GA-641X.

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